A Fast Distributed Asynchronous Newton-Based Optimization Algorithm

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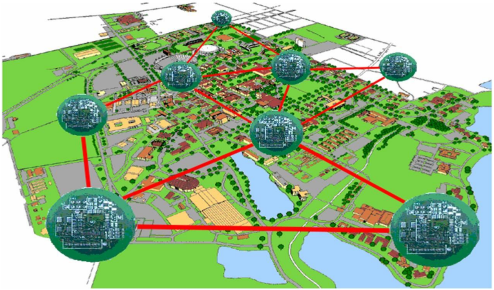
Sensor Network Example

- A network of 3 sensors, supervised passive learning.
- Data is collected at different sensors: input t, output d.



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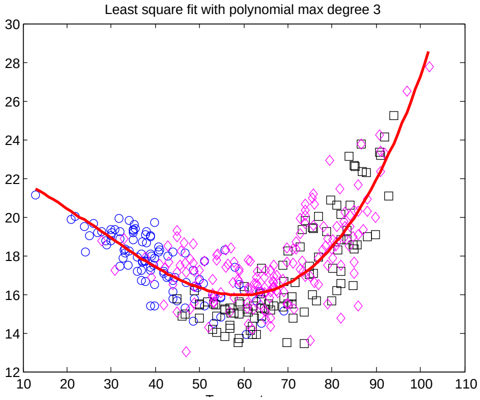


Sensor Network Example

- A network of 3 sensors, supervised passive learning.
- Data is collected at different sensors: input t, output d.

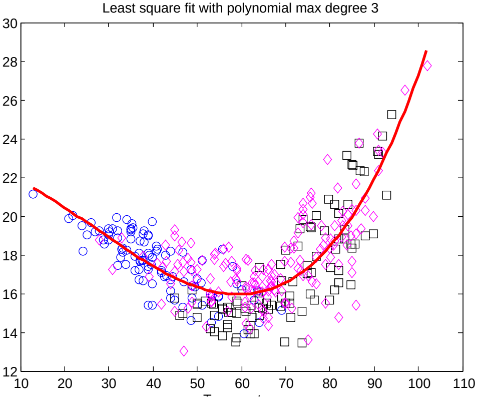
- System goal: a 3rd-degree polynomial model:

$$d(t) = x_3 t^3 + x_2 t^2 + x_1 t + x_0.$$

- System objective:

$$\min_x \quad \sum_{i=1}^{3} ||A_i' x - d_i||_2^2 .$$

where $A_i = [1, t_i, t_i^2, t_i^3]'$ at input data t_i.



Regularized Empirical Loss Minimization Set-up

- System objective: train weight vector x to

$$\min_x \quad \sum_{i=1}^{n-1} L_i(x) + p(x),$$

for some loss function L (on the prediction error) and penalty function p (on the complexity of the model).

  - Example: Least-Absolute Shrinkage and Selection Operator (LASSO):

$$\min_x \quad \sum_{i=1}^{n-1} ||A_i'x - b_i||_2^2 + \lambda \, ||x||_1 \, .$$

  - Other examples from ML estimation, low rank matrix completion, image recovery [Schizas, Ribeiro, Giannakis 08], [Recht, Fazel, Parrilo 10], [Steidl, Teuber, 10]

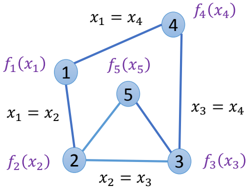
Distributed Multi-agent Optimization

- Connected undirected network of n cooperative agent to solve

$$\min_{x} \quad \sum_{i=1}^{n} f_i(x).$$

- Each function f_i is only locally available to agent i.
- Distributed algorithm: each agent performing computations locally and communicating only to neighbors.[1]
- Introduce local copy x_i for each agent and reformulation to distributed setup

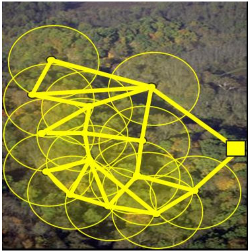
$$\min_{x} \quad \sum_{i=1}^{n} f_i(x_i)$$
$$s.t. \quad x_i = x_j, \quad \text{for} \quad (i,j) \in E.$$



[1]This talk will focus on the case where x is in $\mathbb{R}$. The results generalize to $\mathbb{R}^n$.

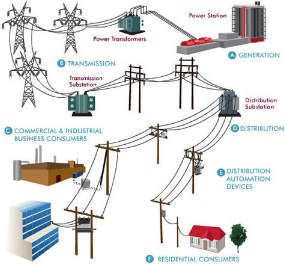
Other Motivations

- Many networks are large-scale, with agents with local information and heterogeneous preferences: call for distributed optimization.
- Most existing distributed optimization algorithms require a <u>central clock</u>.
- This motivated development of asynchronous distributed schemes for control and optimization of multi-agent networked systems.



Parameter estimation in sensor networks

Multi-agent cooperative control and coordination



Smart grid systems

Distributed Multi-agent Optimization

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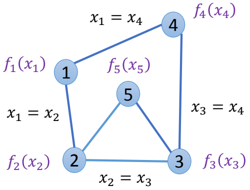
$$\min_{x} \quad \sum_{i=1}^{n} f_i(x).$$

- Each function f_i is only locally available to agent i.
- Distributed algorithm: each agent performing computations locally and communicating only to neighbors.[2]
- Introduce local copy x_i for each agent and reformulation to distributed setup

$$\begin{aligned} \min_{x} \quad & \sum_{i=1}^{n} f_i(x_i) \\ s.t. \quad & x_i = x_j, \quad \text{for} \quad (i,j) \in E. \end{aligned}$$



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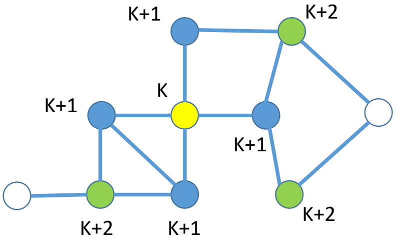
Consensus Based Algorithms

- Each agent has a component of the system objective function and aims to minimize its local cost function.
- Each agent tries to keep its variable equal to those of neighboring agents.
- The basic distributed gradient descent algorithm: each iterate (k), each agent takes a (sub)gradient step and averages with neighbors.

$$x_i(k+1) = \sum_{j=1}^{n} W_{ij}(k)x_j(k) - s(k)d_i(k),$$

where $W_{ij} \geq 0$ weights, $s(k) > 0$ stepsize, $d_i(k)$ subgradient of f_i at $x_i(k)$.



ADMM Based Methods

- Standard ADMM solves a separable problem, where decision variable decomposes into two (linearly coupled) variables:

$$\begin{aligned} \min_{x,y} \quad & f(x) + g(y) \\ \text{s.t.} \quad & Ax + By = c. \end{aligned}$$

- Consider an Augmented Lagrangian function:

$$L_\beta(x, y, p) = f(x) + g(y) - p'(Ax + By - c) + \frac{\beta}{2} ||Ax + By - c||_2^2 .$$

$$\begin{aligned} x^{k+1} &= \text{argmin}_x \quad L_\beta(x, y^k, p^k), \\ y^{k+1} &= \text{argmin}_y \quad L_\beta(x^{k+1}, y, p^k), \\ p^{k+1} &= p^k - \beta(Ax^{k+1} - By^{k+1} - c). \end{aligned}$$

Literature: Synchronous Distributed Optimization Algorithms

- Primal Methods :
  - Consensus and Distributed Gradient Descent Algorithms [Bertsekas, Tsitsiklis 89], [Jadbabaie, Lin, Morse 03], [Blondel, Hendrickx, Olshevsky, Tsitsiklis 05], [Nedić, Ozdaglar 09], [Yuan, Ling, Yin 13], [Jakovetic, Xavier, Moura 14], [Shi, Ling, Wu, Yin 14]

- Dual Methods:
  - Distributed dual averaging and ADMM Algorithms [Bertsekas, Tsitsiklis 89], [Boyd, Parikh, Chu, Peleato, Eckstein 11], [Duchi, Agarwal, Wainwright 12], [Wei, Ozdaglar 12]

- Newton-based Methods:
  - For network flow related problems [Jadbabaie, Ozdaglar, Zargham 09], [Wei, Ozdaglar, Jadbabaie 10], [Liu, Sherali 12]
  - For sum minimization problems: Network Newton method [Mokhtari, Ling, Ribeiro, 15]

Asynchronous vs. Synchronous Algorithms

- Synchronous Algorithms
  - Agents will need to have access to a central coordinator/clock.
  - They must wait for the slowest to finish before proceeding to the next iteration.
- Asynchronous Algorithms
  - Agents become active randomly in time and update using delayed/partial and local information.
  - There is no need for a central coordinator.
  - Partially asynchronous: bounded delay.
  - Totally asynchronous: infinitely often update for each agent.

Literature: Asynchronous Distributed Optimization Algorithms

- Primal Methods :
  - Gossip-Based Algorithms [Boyd, Ghosh, Prabhakar, Shah 06], [Ram, Nedić, Veeravalli 09]
  - Broadcast-Based Algorithms [Nedić 11]
  - Coordinate-Based Algorithms [Peng, Xu, Yan, Yin 16], [Hannah, Yin 16]
- Primal-Dual Methods:
  - Coordinate Descent Algorithms [Binachi, Hachem, Iutzeler 15]
  - Asynchronous Distributed ADMM [Wei, Ozdeglar 13], [Chang, Hong, Liao, Wang 16]
- Quasi Newton Methods: [Eisen, Mokhtari, Ribeiro 16], [Bajovic, Jakovetic, Krejic, Jerinkic 17]
- These algorithms can be shown to achieve:
  - Totally asynchronous: sublinear linear rates (linear if similar delays).

This Talk

- We present an asynchronous distributed network Newton algorithm for multi-agent optimization with “nice” objective functions.
- Asynchronous algorithm and distributed implementation
    - Hessian approximation method
    - Algorithm and implementation
- Convergence analysis
    - Almost sure (global) convergence
    - Global linear convergence
    - Local quadratic convergence

Problem Formulation

We will consider minimizing the penalized version of the consensus problem over a connected undirected static network of n agents. For $x = [x_i]_i$,

$$\min_x F(x) = \quad \frac{1}{2} x^T (I - W) x + \alpha \sum_{i=1}^{n} f_i(x_i).$$

- "Nice": The local objective functions $f_i(x)$ are convex, twice continuously differentiable. Hessian matrices have bounded eigenvalues and are L-Lipschitz continuous.
- Each agent i knows f_i and updates x_i.
- Matrix W is a symmetric doubly stochastic consensus/weight matrix: for all i

$$W = W', \quad \sum_{j=1}^{n} W_{ij} = 1 \ , \quad 0 < W_{ii} < 1.$$

- Matrix W represents the network topology: $W_{ij} \neq 0 \quad iff \quad (i,j) \in E$ for $i \neq j$.
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Standard Newton's Algorithm

- Our asynchronous method is based on Newton's algorithm for unconstrained problem with iteration

$$x(t+1) = x(t) + \varepsilon d(t),$$

  - ε is some positive stepsize and $d(t)$ is the Newton direction.
  - The Newton's direction is $d(t) = H(t)^{-1}g(t)$ with
    $H(t) = \nabla^2 F(x(t)) = I - W + \alpha G(t), \quad \text{where} \quad G_{ii}(t) = \nabla^2 f_i(x_i(t)).$
    $g_i(t) = \nabla_i F(x(t)) = [(I - W)x(t)]_i + \alpha \nabla f_i(x_i(t)).$

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- Asynchronous network Newton uses the matrix splitting techniques and truncated Taylor expansion to approximate the Hessian inverse in a distributed manner.

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Background on Hessian Approximation

- Hessian matrix can be splitted as $H(t) = I - W + \alpha G(t) = D(t) - B$ with

$$D(t) = \alpha G(t) + 2(I - W_d), \quad B = I - 2W_d + W$$

  where W_d is a diagonal matrix with $[W_d]_{ii} = W_{ii}$.
- $D(t)$ is positive definite and thus invertible.
- We can write $H(t)^{-1}$ as

$$H(t)^{-1} = D(t)^{-1/2}(I - D(t)^{-1/2}BD(t)^{-1/2})^{-1}D(t)^{-1/2}$$

- We also have, if $\rho(A) < 1$, then $(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$.
- From [Mokhtari, Ling, and Ribeiro 15] $\rho(D(t)^{-1/2}BD(t)^{-1/2}) < 1$. Then by finite truncation, we have Hessian inverse approximation

$$\hat{H}(t)^{-1} = D(t)^{-1/2}\left[I + D(t)^{-1/2}BD(t)^{-1/2}\right]D(t)^{-1/2}.$$

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Distributed Computation of Approximate Hessian Inverse Matrix

$$\hat{H}(t)^{-1} = D(t)^{-1/2}\left[I + D(t)^{-1/2} B D(t)^{-1/2}\right] D(t)^{-1/2}.$$

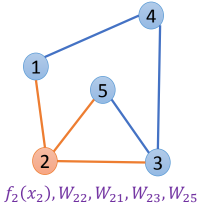
- Each agent i has local access to f_i and W_{ii} and to W_{ij} for all neighbors j in $\mathcal{N}(i)$.
- At iteration t

  a. Each agent i computes the $i'th$ diagonal element of $D(t)$,

  $$D_{ii}(t) = \alpha \nabla^2 f_i(x_i(t)) + 2(1 - W_{ii}),$$

  b. Each agent i computes B_{ii} and B_{ij} for all neighbors j.

  $$B_{ii} = 1 - 2W_{ii} + W_{ii} = 1 - W_{ii}, \quad B_{ij} = W_{ij},$$



Multiplication of matrix B corresponds to communicating with immediate neighbors, which can also be carried out locally. The Newton's direction approximation would be

$$d(t) = \hat{H}(t)^{-1} g(t).$$

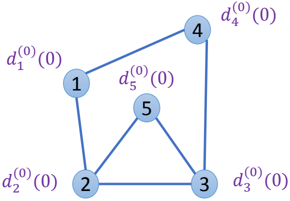
Asynchronous Network Newton

- Each agent i is associated with a local clock that ticks with probability p_i, $(0 < \pi \leq p_i \leq \Pi < 1, \sum_{i=1}^{n} p_i = 1)$.
- Whenever the clock ticks, an agent is active and updates its local Newton direction.
- Each agent's stepsize is inversely proportional to its activation probability.
- The active agent finishes updating before another activation happens.
- Whenever any agent is active the iteration counter is increased by 1.

Asynchronous Network Newton Algorithm

Initialization: For $i = 1, 2, ..., n$, each agent i sets $x_i(0) = 0$, computes $D_{ii}(0)$, $g_i(0)$, $d_i^{(0)}(0)$, B_{ii}, and B_{ij}:

$$D_{ii}(0) = \alpha\nabla^2 f_i(x_i(0)) + 2(1 - W_{ii}),$$
$$g_i(0) = (1 - W_{ii})x_i(0) + \alpha\nabla f_i(x_i(0)),$$
$$d_i^{(0)}(0) = -D_{ii}(0)^{-1} g_i(0),$$
$$B_{ii} = 1 - W_{ii}, \quad B_{ij} = W_{ij},$$



and broadcasts $d_i^{(0)}(0)$ to all neighbors, stores received $d_j^{(0)}$, x_j values from neighbors.

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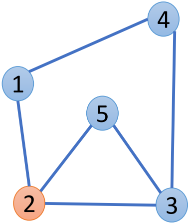
a For $t = 1, 2, \ldots$, an agent i is active according to its local clock with probability p_i, computes its local Newton's direction and updates its local iterate

$$g_i(t-1) = (1 - W_{ii})x_i(t-1) + \alpha \nabla f_i(x_i(t-1)) - \sum_{j \in \mathcal{N}_i} W_{ij} x_j(t-1)$$

$$d_i^{(0)}(t-1) = -D_{ii}(t-1)^{-1} g_i(t-1)$$



$g_2(t-1), d_2^{(0)}(t-1)$

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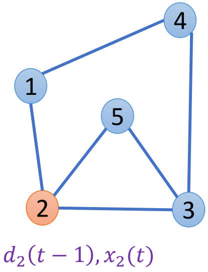
$$d_i(t-1) = D_{ii}(t-1)^{-1}\left[B_{ii}d_i^{(0)}(t-1) - g_i(t-1) + \sum_{j\in\mathcal{N}_i} B_{ij}d_j^{(0)}(t-1)\right]$$

$$x_i(t) = x_i(t-1) + \frac{\varepsilon}{p_i}d_i(t-1)$$



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b Active agent updates $D_{ii}(t), g_i(t), d_i^{(0)}(t)$

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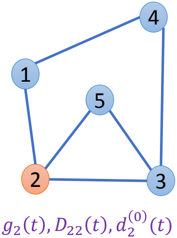

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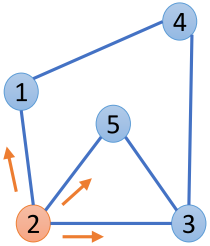
b Active agent updates $D_{ii}(t), g_i(t), d_i^{(0)}(t)$

c Active agent i broadcasts $d_i^{(0)}(t), x_i(t)$ to its neighbors who passively listen and store received most updated values.



Almost Sure Convergence and Global Linear Rate

Theorem

When the stepsize parameter satisfies $0 < \varepsilon \leq 2\pi \left(\frac{\lambda}{\Lambda}\right)^2$*, the sequence* $\{F(x(t))\}$ *converges to its optimal value* F^* *almost surely.*

If $0 < \varepsilon \leq \min\left\{\frac{1}{2}, 2\pi \left(\frac{\lambda}{\Lambda}\right)^2\right\}$*, then* $\{F(x(t))\}$ *and* $\{x(t)\}$ *converge linearly in expectation, i.e.,*

$$\mathbb{E}\left[F(x(t)) - F^*\right] \leq (1-\beta)^t \left[F(x(0)) - F^*\right],$$

$$\mathbb{E}\left[\,||x(t) - x^*||\,\right] \leq \left(\frac{2\left(F(x(0)) - F^*\right)}{\alpha m}\right)^{1/2} \left((1-\beta)^{1/2}\right)^t.$$

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$$\mathbb{E}\left[F(x(t)) - F^*\right] \leq (1-\beta)^t \left[F(x(0)) - F^*\right],$$

$$\mathbb{E}\left[\|x(t) - x^*\|\right] \leq \left(\frac{2\left(F(x(0)) - F^*\right)}{\alpha m}\right)^{1/2} \left((1-\beta)^{1/2}\right)^t.$$

Constants: $\Lambda = \frac{1+\rho}{2(1-\Delta)+\alpha m}$, $\lambda = \frac{1}{2(1-\delta)+\alpha M}$, $0 < \beta = \frac{\alpha m \varepsilon (2\pi\lambda^2 - \varepsilon\Lambda^2)}{\lambda\pi} < 1$, $\pi = \min_i p_i$, $\delta = \min_i W_{ii}$, $\Delta = \max_i W_{ii}$, $mI \preceq \nabla^2 f_i(x_i) \preceq MI$, $\rho = 2(1-\delta)(2(1-\delta)+\alpha m)$

Local Superlinear Convergence

Lemma

For stepsize with $0 < \varepsilon \leq min \left\{ \frac{1}{2}, 2\pi \left(\frac{\lambda}{\Lambda}\right)^2 \right\}$, *we have*

$$\mathbb{E}\left[\left|\left| D(t-1)^{1/2}(x(t) - x^*) \right|\right| \right] \leq \Gamma_1 \left(\mathbb{E}\left[\left|\left| D(t-2)^{1/2}(x(t-1) - x^*) \right|\right| \right] \right)^2$$

$$+ \Gamma(t)\mathbb{E}\left[\left|\left| D(t-2)^{1/2}(x(t-1) - x^*) \right|\right| \right].$$

Local Superlinear Convergence

Lemma

For stepsize with $0 < \varepsilon \leq min \left\{\frac{1}{2}, 2\pi\left(\frac{\lambda}{\Lambda}\right)^2\right\}$, *we have*

$$\mathbb{E}\left[\left|\left|D(t-1)^{1/2}(x(t)-x^*)\right|\right|\right] \leq \Gamma_1\left(\mathbb{E}\left[\left|\left|D(t-2)^{1/2}(x(t-1)-x^*)\right|\right|\right]\right)^2$$
$$+\Gamma(t)\mathbb{E}\left[\left|\left|D(t-2)^{1/2}(x(t-1)-x^*)\right|\right|\right].$$

Constants: $\Gamma_1 = \frac{\left(2(1-\delta)+\alpha M\right)^{1/2}\alpha L\varepsilon\Lambda}{2\pi^2\left(2(1-\Delta)+\alpha m\right)}$ and $\Gamma(t) = C_1\left(1 + C_3(1-\beta)^{\frac{t-2}{4}}\right)$ with

$C_1 = \left(1+\varepsilon \max\left\{\frac{\varepsilon}{\pi} - 2, \frac{\varepsilon(1-\rho^2)^2}{\pi} - 2(1-\rho^2)\right\}\right)^{1/2} < 1$, $C_2 = \left(\frac{\varepsilon\alpha L\Lambda}{\pi\left(2(1-\Delta)+\alpha m\right)}\right)^{1/2}$, and

$C_3 = C_2\left(\frac{2}{\lambda\pi^2}\left(F(x(0)) - F^*\right)\right)^{1/4}$.

Local Superlinear Convergence

Theorem

For all t with

$$t > \frac{4 \ln \frac{1-C_1}{C_3 C_1}}{\ln(1-\beta)} + 2,$$

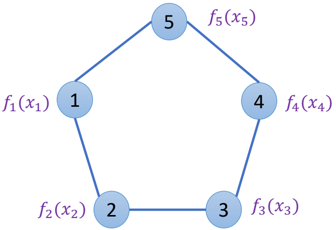
we have $\Gamma(t) < 1$ and there exists $0 < \theta < \frac{1-\Gamma(t)}{\Gamma_1 \Gamma(t)}$, such that the sequence $\mathbb{E}\left[\left\| D(t-1)^{1/2}\left(x(t) - x^*\right)\right\|\right]$ decreases with a quadratic rate in expectation in this interval.

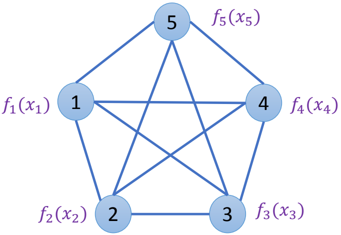
This neighborhood is also characterized by

$$\theta \Gamma(t) \leq \mathbb{E}\left[\left\| D(t-1)^{1/2}\left(x(t) - x^*\right)\right\|\right] < \frac{\theta}{\theta \Gamma_1 + 1},$$

Simulation Results

- Asynchronous network Newton is compared against asynchronous ADMM [Wei, Ozdeglar 13] and asynchronous gossip [Ram, Nedić, Veeravalli 09] algorithms.
- Tested on networks of 5 agents with complete and ring underlying graphs.
- Tested on quadratic and non-quadratic objective functions.
- Asynchronous network Newton outperforms the other two algorithms, which is expected due to the local quadratic rate.

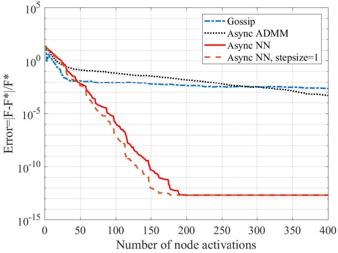




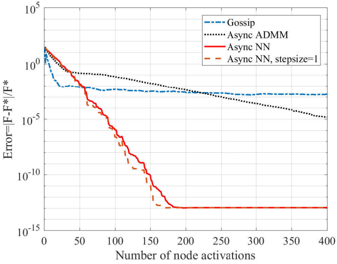
Quadratic Objective Functions

- Objective function at agent i: $f_i(x_i) = (x - i)^2$
- Minimum activation probability: $\pi = \frac{2}{15}$
- Stepsize parameter for Async NN: $\varepsilon = \frac{2}{15}$

Complete Graph



Ring Graph



Regularized Logistic Regression

- Data classification for K training samples that are uniformly distributed over $n = 5$ agents in a network.
- Each agent i has access to $k_i = \lfloor \frac{K}{n} \rfloor$ data points.
- u_{ij} and v_{ij}, $j \in \{1, 2, ..., k_i\}$ are the feature vector and the label for the data point j associated with agent i.

$$\min_x f(x) = \frac{\upsilon}{2} ||x||^2 + \frac{1}{K} \sum_{i=1}^{n} \sum_{j=1}^{k_i} \log \left[1 + \exp(-v_{ij} u_{ij} x)\right],$$

$$f_i(x) = \frac{\upsilon}{2n} ||x||^2 + \frac{1}{K} \sum_{j=1}^{k_i} \log \left[1 + \exp(-v_{ij} u_{ij} x)\right].$$

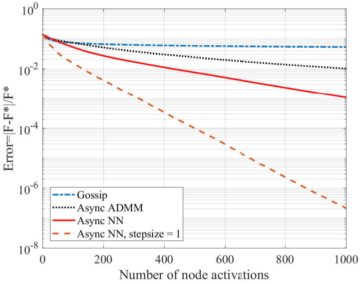
- Tested on "Pima Indian Diabetes" data set with 768 data points, feature vector of size 8, and a label which is either 1 or -1.

Regularized Logistic Regression

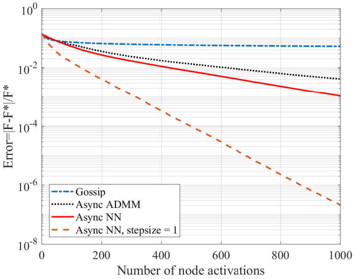
$$f_i(x) = \frac{\upsilon}{2n} ||x||^2 + \frac{1}{K} \sum_{j=1}^{k_i} \log\left[1 + \exp(-v_{ij} u_{ij} x)\right].$$

- Minimum activation probability: $\frac{2}{15}$
- Stepsize parameter for Async NN: $\varepsilon = 0.043$

Complete Graph



Ring Graph



Conclusions and Future Directions

- Asynchronous distributed network Newton algorithm uses matrix splitting techniques to approximate the Hessian inverse and compute Newton step.
- This algorithm converges almost surely with in expectation global linear rate of convergence.
- Asynchronous network Newton achieves a local quadratic convergence rate (in expectation) to a neighborhood of the optimum.
- Simulation results show the convergence speed improvement of the asynchronous network Newton compared to asynchronous ADMM and asynchronous gossip algorithm.

- Future directions:
  - Dynamic graph.
  - Larger stepsize rules.